Solutions to be posted by next weekend 03/02/2018.

The purpose of this practice is to make you fast with manipulations of operators.

1. Here are some examples of operators commonly used in quantum mechanics.

   i Translation $\hat{T}_a \equiv e^{a\frac{d}{dx}}$. Work out and show what will be the effect of this operator on a trial function $\psi(x)$.

   ii Scaling or dilation $\hat{S}_a \equiv e^{a\hat{x}\frac{\hat{d}}{dx}}$. Work out and show what will be the effect of this operator on a trial function $\psi(x)$.

   iii Projection: $\hat{h}(x)$, $h(x) = 0$ when $x < 0$ and $h(x) = 1$ when $x \geq 0$.

   iv $\hat{A}_s\psi(x) \equiv \frac{1}{\sqrt{2}}[\psi(x) + \psi(-x)]$

   (a) Verify that each of the above operators is linear.

   (b) Draw qualitative plots of the action of each of the operators on a purely position dependent wave function. Take any form of the wave function that qualifies to be square integrable.

   (c) Show that the last two operators listed, satisfy the relation $\hat{A}^2\psi = \hat{A}\psi$

2. Does the translation operator given in question 1 preserve the normalization of the wave function? A variant of this operator is given as:

$$e^{a\left(\hat{x}\frac{\hat{d}}{dx}+\frac{\hat{d}}{dx}\hat{x}\right)}\psi(x) = ?$$

   What is the action of this operator? Will this operator preserve the normalization of the wave function?

3. If $\hat{A}$ and $\hat{B}$ are two operators that commute with their commutator, prove that, for a positive n,

$$\left[\hat{A}, \hat{B}^n\right] = n\hat{B}^{n-1}\left[\hat{A}, \hat{B}\right]$$
$$\left[\hat{A}^n, \hat{B}\right] = n\hat{A}^{n-1}\left[\hat{A}, \hat{B}\right]$$

   What is this process similar to? Take a special case of this, where $\hat{A} \equiv \hat{x}$ and $\hat{B} \equiv \hat{p}_x$. What is the commutator $\left[\hat{p}_x, \hat{f}(x)\right]$, if $\hat{f}(x)$ can be expanded in a power series of $\hat{x}$?

4. Evaluate the following commutators:

   (a) $\left[\hat{A}, \left[\hat{B}, \hat{C}\right]\hat{D}\right]$

   (b) Angular momentum components, $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$. Give the commutators $\left[\hat{L}_x, \hat{L}_y\right]$, $\left[\hat{L}_y, \hat{L}_z\right]$ and $\left[\hat{L}_z, \hat{L}_x\right]$. Is there something interesting that you see in all the three?

   (c) $\left[\hat{\vec{L}}^2, \hat{L}_z\right]$

   (d) $\left[\hat{T}, \hat{V}(x,y,z)\right]$ where $\hat{T}$ is the kinetic energy operator for a particle moving in 3 dimensional cartesian space and $\hat{V}(x,y,z)$ is the potential energy operator which is multiplicative.

   (e) $\left[\hat{H}, \hat{T}\right]$

   (f) $\left[\hat{H}, \hat{V}(x,y,z)\right]$

   (g) $\left[\hat{H}, \hat{\vec{r}}\right]$

   (h) $\left[\hat{H}, \hat{\vec{p}}\right]$

(i) $\left[\vec{r}\cdot\hat{\vec{p}},\hat{H}\right]$

5. The *distributivity* of commutators is given by:

$$\left[\hat{A}\hat{B},\hat{C}\right]=\hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B}$$
$$\left[\hat{A},\hat{B}\hat{C}\right]=\left[\hat{A},\hat{B}\right]\hat{C}+\hat{B}\left[\hat{A},\hat{C}\right]$$

These you had verified in the tutorial session 1. Using a repeated application of one or both of these relations show that:

$$\left[\hat{A},\hat{B}^n\right] = \sum_{j=0}^{n-1}\hat{B}^j\left[\hat{A},\hat{B}\right]\hat{B}^{n-j-1}$$
$$\left[\hat{A}^n,\hat{B}\right] = \sum_{j=0}^{n-1}\hat{A}^{n-j-1}\left[\hat{A},\hat{B}\right]\hat{A}^j$$

6. Evaluate the commutators $[\hat{x}^n,\hat{p}_x]$,$[\hat{x},\hat{p}_x^n]$,$[f(\hat{x}),\hat{p}_x]$ and $\left[\hat{\vec{p}},f\left(\hat{\vec{r}}\right)\right]$.

Warning: The following are tough problems and very mathematical, hence **optional**.

7. Consider the operator $\hat{f}(\lambda)=e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$ where λ is a real number.
   (a) Convince yourself that $\hat{f}(\lambda)$ is indeed an operator.
   (b) Write down a Taylor series expansion for $\hat{f}(\lambda)$ about $\lambda=0$.
   (c) What is $\frac{d\hat{f}}{d\lambda}$? Can this be written as a commutator? If yes, then what is the commutator?
   (d) What is $\frac{d^2\hat{f}}{d\lambda^2}$? Can this be written as a commutator of a commutator? If yes, then what is the commutator?
   (e) What is $\hat{f}(0)$, *viz.* $\hat{f}(\lambda)\Big|_{\lambda=0}$?
   (f) Using all the answers from above what is the identity that you get for $e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$?

8. What happens to $e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$ if the commutator $\left[\hat{A},\hat{B}\right]=\mu\hat{1}$, where μ is a constant?

9. If $\left[\hat{A},\hat{B}\right]=\gamma\hat{B}$, where γ is a constant what is $e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$?

10. For the product $\hat{G}(\lambda)=e^{\lambda\hat{A}}e^{\lambda\hat{B}}$ prove that:

$$\frac{d\hat{G}}{d\lambda} = \left(\hat{A}+\hat{B}+\frac{\lambda}{1!}\left[\hat{A},\hat{B}\right]+\frac{\lambda^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]+\dots\right)\hat{G}$$
$$= \hat{G}\left(\hat{A}+\hat{B}+\frac{\lambda}{1!}\left[\hat{A},\hat{B}\right]+\frac{\lambda^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]+\dots\right)$$